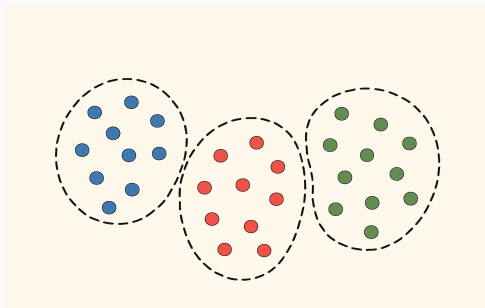


Dynamic k -Center Clustering

Antonis Skarlatos



Introduction



Introduction



Definition (k -Center)

Input: Point set P , integer $k \geq 1$

Output: Subset $S \subseteq P$:

$$|S| \leq k$$

$\max_{p \in P} \text{dist}(p, S)$ is minimized

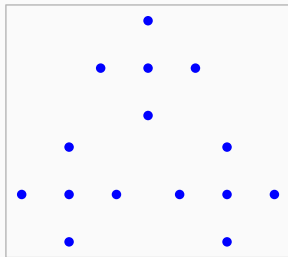
k -Center Clustering Problem

Definition (k -Center)

Input: Point set P , integer $k \geq 1$

Output: Subset $S \subseteq P$:

$k = 3$



k -Center Clustering Problem

Definition (k -Center)

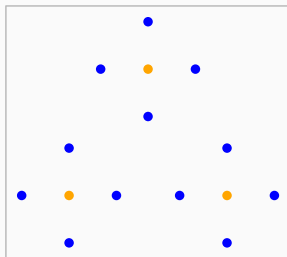
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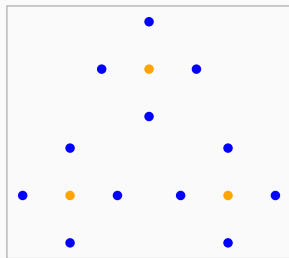
$\max_{p \in P} \text{dist}(p, S)$ is minimized

$$\text{dist}(p, S) := \min_{q \in S} \text{dist}(p, q)$$

Arbitrary Metric Space \Rightarrow

Triangle Inequality

$k = 3$



k -Center Clustering Problem

Definition (k -Center)

Input: Point set P , integer $k \geq 1$

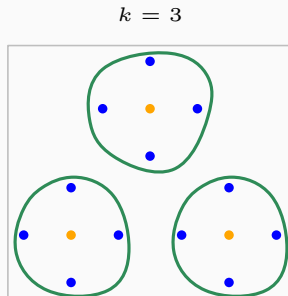
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$\max_{p \in P} \text{dist}(p, S)$ is minimized

Assign each **point** to **closest center**

\Rightarrow **clusters**



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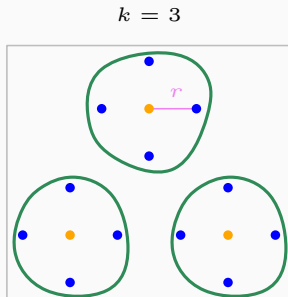
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Radius: Max distance of any **point** to **closest center**



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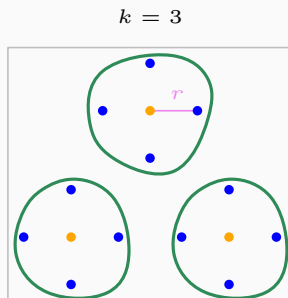
$\max_{p \in P} \text{dist}(p, S)$ is minimized

Assign each **point** to **closest center**

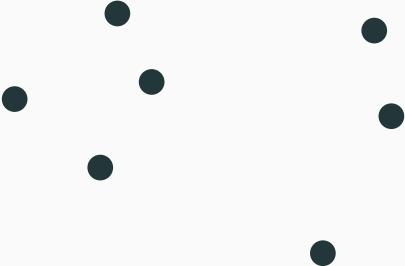
\Rightarrow **clusters**

Radius: Max distance of any **point** to **closest center**

NP-hard to approximate within a factor of $2 - \epsilon$



Dynamic Input



Dynamic Input



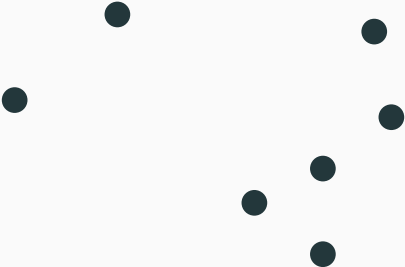
Dynamic Input



Dynamic Input

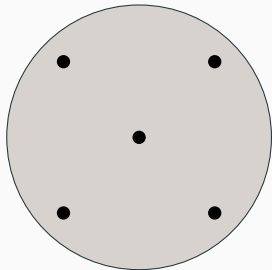


Dynamic Input

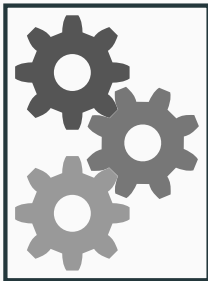


Dynamic Setting

Dynamic point set P



Dynamic algorithm

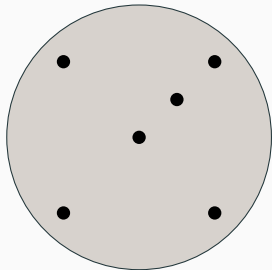


k -Center solution

$$S = \{c_1, \dots, c_k\}$$

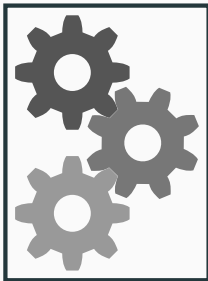
Dynamic Setting

Dynamic point set P



Adversary inserts
new point

Dynamic algorithm

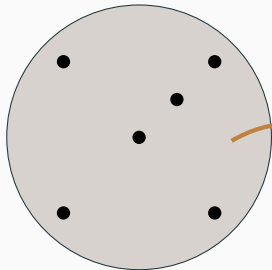


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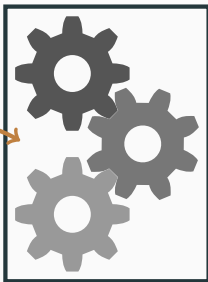
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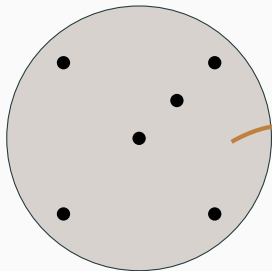


k -Center solution

$$S = \{c_1, \dots, c_k\}$$

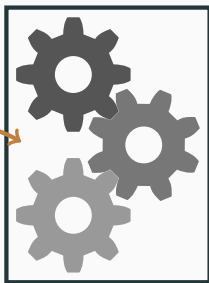
Dynamic Setting

Dynamic point set P



Adversary inserts
new point

Dynamic algorithm



k -Center solution

$$S = \{\hat{c}_1, \dots, \hat{c}_k\}$$

Algorithm updates
the k centers

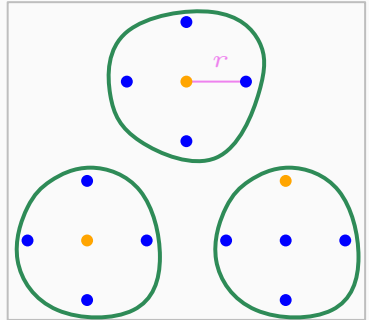
Dynamic k -Center Clustering

Fully Dynamic k -Center

Updates: Insert/Delete point

$k = 3$

r is the cost

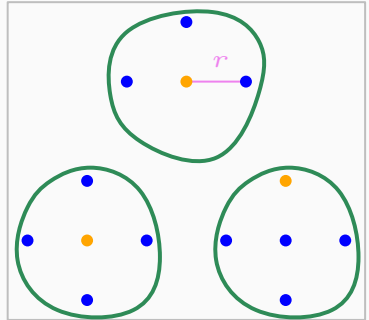


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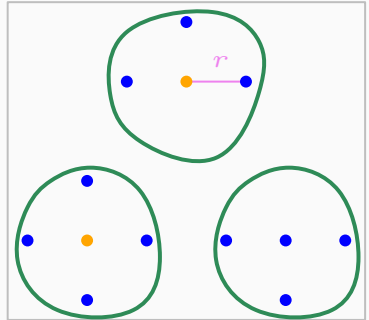


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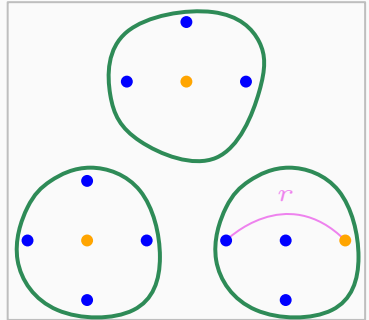


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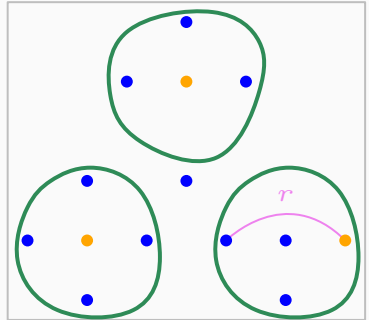


Fully Dynamic k -Center

Updates: Insert/Delete point

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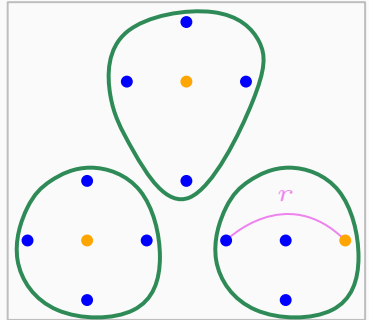


Fully Dynamic k -Center

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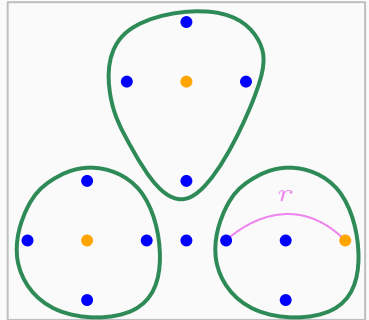


Fully Dynamic k -Center

Updates: **Insert**/**Delete** point

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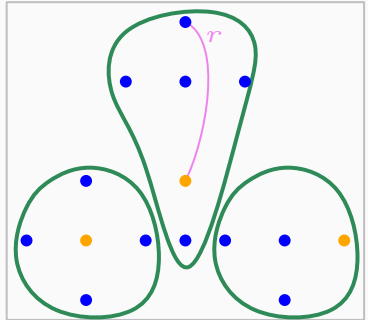


Fully Dynamic k -Center

Updates: Insert/Delete point

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Fully Dynamic k -Center

Updates: **Insert/Delete** point

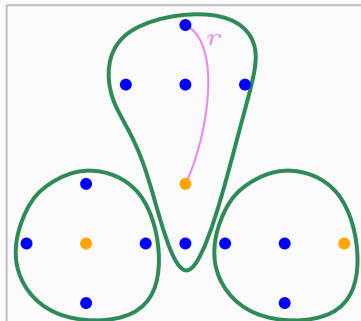
Optimize:

$O(1)$ approximation

$O(1)$ recourse

$k = 3$

r is the cost



Recourse: the number of changes to **centers**

Fully Dynamic k -Center

Updates: Insert/Delete point

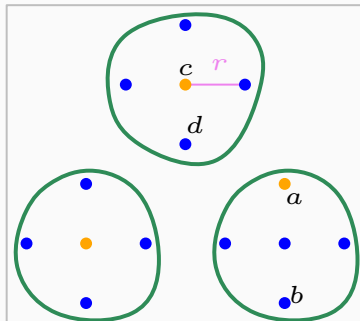
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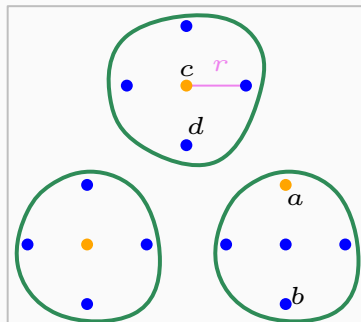
Optimize:

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r is the cost



Adversary deletes a

Recourse: the number of changes to **centers**

Fully Dynamic k -Center

Updates: Insert/Delete point

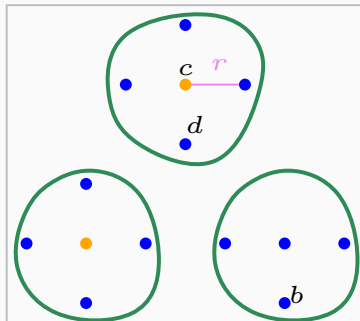
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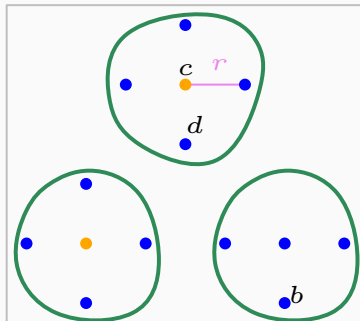
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Remove a from centers

Add b to centers

Recourse: the number of changes to centers

Fully Dynamic k -Center

Updates: Insert/Delete point

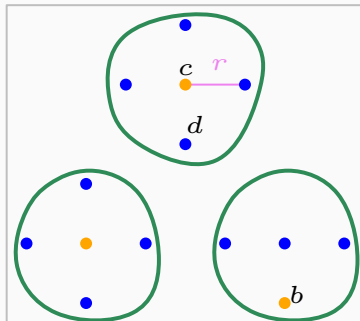
Optimize:

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Recourse: the number of changes to centers

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Updates: **Insert/Delete** point

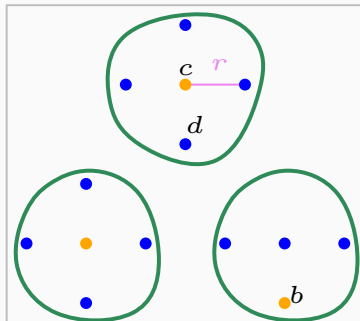
Optimize:

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$k = 3$

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Remove c from centers

Add d to centers

Recourse: the number of changes to **centers**

Fully Dynamic k -Center

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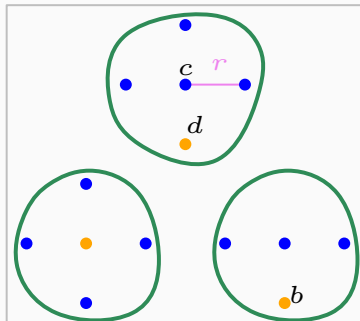
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Fully Dynamic k -Center

Updates: Insert/Delete point

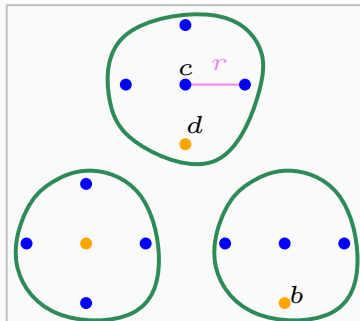
Optimize:

$O(1)$ approximation

$O(1)$ recourse

$k = 3$

r is the cost



recourse is 2

Recourse: the number of changes to centers

Setup for k -Center Algorithm

Certificate for k -Center Clustering

Lemma ([Hochbaum, Shmoys '86])

If $k + 1$ points in P whose pairwise distances $\geq r$

Certificate for k -Center Clustering

Lemma ([Hochbaum, Shmoys '86])

If $k + 1$ points in P whose pairwise distances $\geq r$

then $\frac{r}{2} \leq OPT$

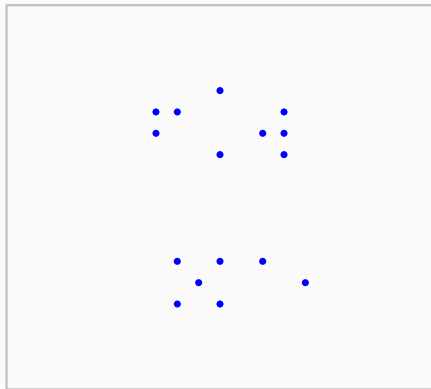
Certificate for k -Center Clustering

Lemma ([Hochbaum, Shmoys '86])

If $k + 1$ points in P whose pairwise distances $\geq r$

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$k = 2$



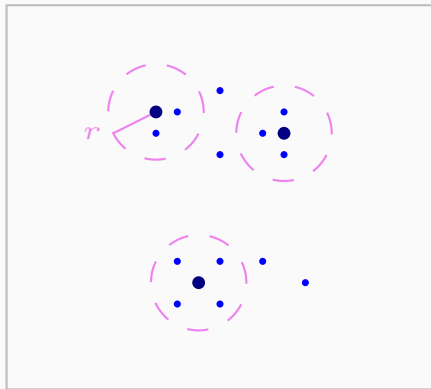
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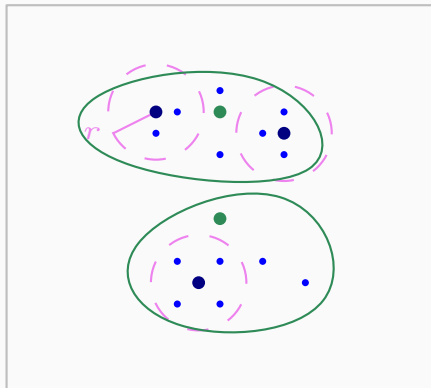
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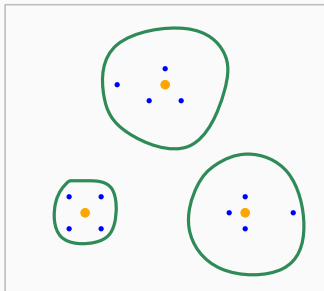


Approximation Ratio

Invariants for fixed constants $\alpha \geq 1$

Invariant 1: $|S| = k$

$k = 3$



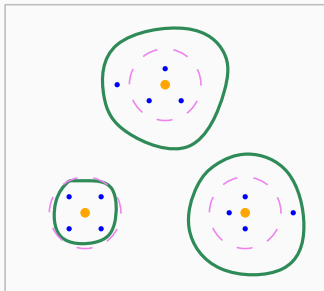
Approximation Ratio

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Invariant 2: $\forall q_1, q_2 \in S : \text{dist}(q_1, q_2) \geq r$

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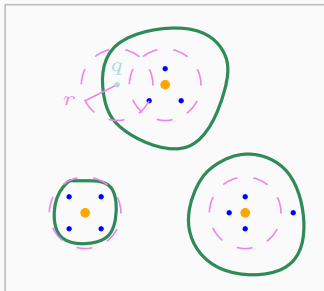
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 $\exists q \in P \setminus S : \text{dist}(q, S) \geq r$

$k = 3$



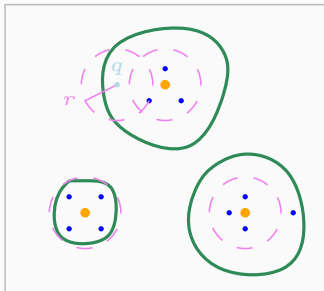
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$k = 3$



Invariant 1 and Invariant 2

$$r \leq 2 \cdot \text{OPT}$$

Approximation Ratio

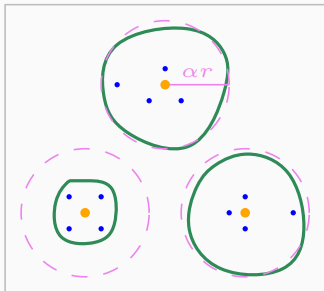
Invariants for fixed constants $\alpha \geq 1$

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Invariant 3: $\forall p \in P : \text{dist}(p, S) \leq \alpha \cdot r$

$k = 3$



Invariant 1 and Invariant 2

$$r \leq 2 \cdot OPT$$

Approximation Ratio

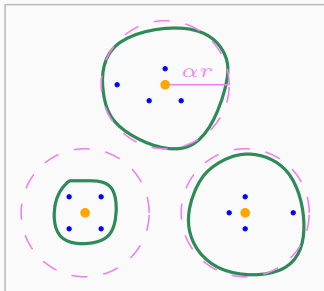
Invariants for fixed constants $\alpha \geq 1$

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$k = 3$



Invariant 1 and Invariant 2

$$r \leq 2 \cdot OPT$$

Together with Invariant 3

S is (2α) -approximate

Incremental Setting

Point Insertion

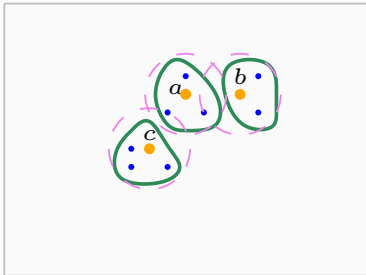
Invariants

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$$k = 3$$



Point Insertion

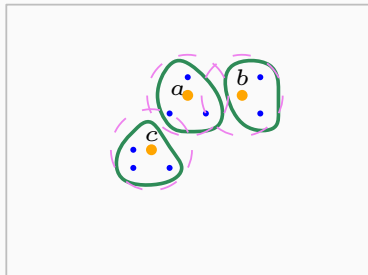
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$$k = 3$$



Observation: After a point insertion, Invariant 3 could be violated

Point Insertion

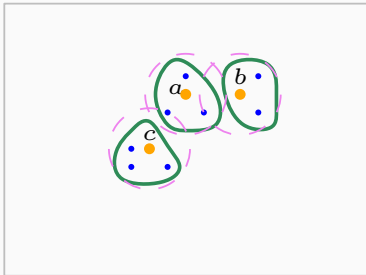
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Invariant 3: $\forall p \in P : \text{dist}(p, S) \leq r$

$k = 3$



$\text{dist}(a, b), \text{dist}(a, c), \text{dist}(b, c) \geq r$

Point Insertion

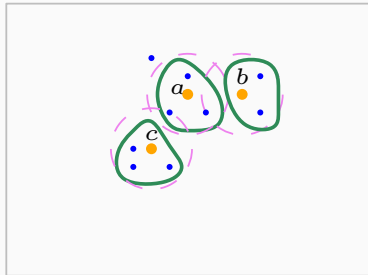
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Invariant 3: $\forall p \in P : \text{dist}(p, S) \leq r$

$k = 3$



$\text{dist}(a, b), \text{dist}(a, c), \text{dist}(b, c) \geq r$

$\text{dist}(p^+, S) > r$

Point Insertion

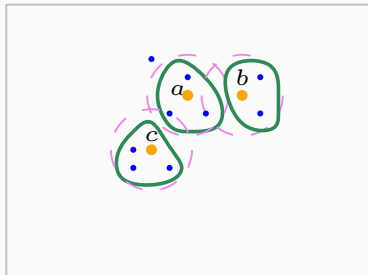
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$k = 3$



$\text{dist}(a, b), \text{dist}(a, c), \text{dist}(b, c) \geq r$

$\text{dist}(p^+, S) > r$

Double r until **Inv 3** is satisfied or **Inv 2** is tight

Point Insertion

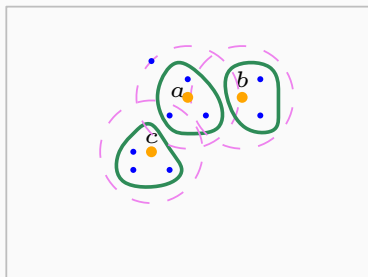
Invariants

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Invariant 2: $\forall q_1, q_2 \in S : \text{dist}(q_1, q_2) \geq r$
 $\exists q \in P \setminus S : \text{dist}(q, S) \geq r$

Invariant 3: $\forall p \in P : \text{dist}(p, S) \leq r$

$k = 3$



$\text{dist}(a, b), \text{dist}(a, c), \text{dist}(b, c) \geq r$

$\frac{r}{2} < \text{dist}(p^+, S) \leq r$

Double r

Double r until **Inv 3** is satisfied or **Inv 2** is tight

Point Insertion

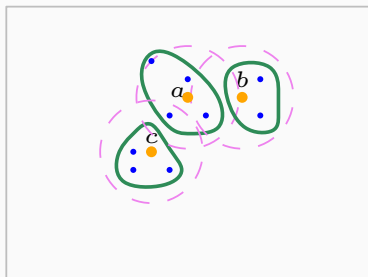
Invariants

Invariant 1: $|S| = k$

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Invariant 3: $\forall p \in P : \text{dist}(p, S) \leq r$

$k = 3$



$\text{dist}(a, b), \text{dist}(a, c), \text{dist}(b, c) \geq r$

$\frac{r}{2} < \text{dist}(p^+, S) \leq r$

Double r

Add p^+ to C_a

Double r until **Inv 3** is satisfied or **Inv 2** is tight

Point Insertion

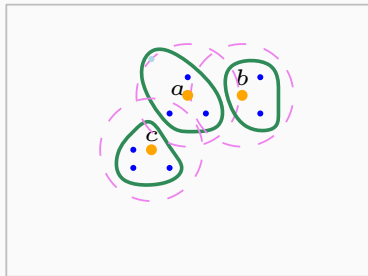
Invariants

Invariant 1: $|S| = k$

Invariant 2: $\forall q_1, q_2 \in S : \text{dist}(q_1, q_2) \geq \frac{r}{2}$
 $\exists q \in P \setminus S : \text{dist}(q, S) \geq \frac{r}{2}$

Invariant 3: $\forall p \in P : \text{dist}(p, S) \leq r$

$k = 3$



$\text{dist}(a, b), \text{dist}(a, c), \text{dist}(b, c) \geq r$

$\frac{r}{2} < \text{dist}(p^+, S) \leq r$

Double r

Add p^+ to C_a , $q = p^+$

Double r until **Inv 3** is satisfied or **Inv 2** is tight

Point Insertion

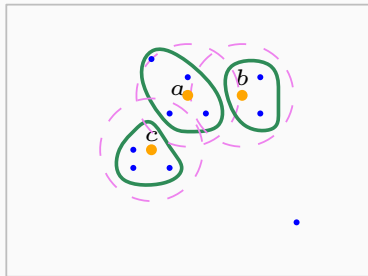
Invariants

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 $\exists q \in P \setminus S : \text{dist}(q, S) \geq \frac{r}{2}$

Invariant 3: $\forall p \in P : \text{dist}(p, S) \leq r$

$k = 3$



$\text{dist}(a, b), \text{dist}(a, c), \text{dist}(b, c) \geq r$

$\text{dist}(p^+, S) > r$

Double r until **Inv 3** is satisfied or **Inv 2** is tight

Point Insertion

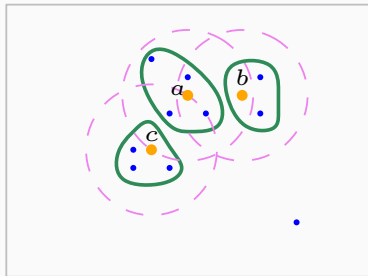
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$k = 3$



$\text{dist}(a, b), \text{dist}(a, c), \text{dist}(b, c) \geq \frac{r}{2}$

$\text{dist}(p^+, S) > r$

Double r

Double r until **Inv 3** is satisfied or **Inv 2** is tight

Point Insertion

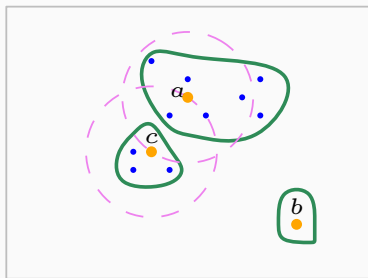
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Invariant 2: $\forall q_1, q_2 \in S : \text{dist}(q_1, q_2) \geq \frac{r}{2}$
 $\exists q \in P \setminus S : \text{dist}(q, S) \geq \frac{r}{2}$

Invariant 3: $\forall p \in P : \text{dist}(p, S) \leq 2r$

$k = 3$



$\text{dist}(a, b), \text{dist}(a, c), \text{dist}(b, c) \geq r$

$\text{dist}(p^+, S) \leq r$

Double r

Add C_b to C_a , C_a becomes *extended*

Double r until **Inv 3** is satisfied or **Inv 2** is tight

Point Insertion

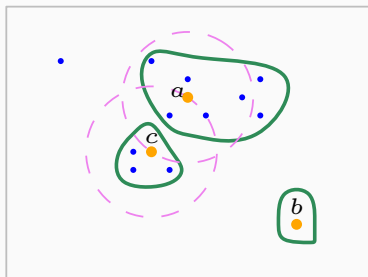
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$\text{dist}(a, b), \text{dist}(a, c), \text{dist}(b, c) \geq r$

$\text{dist}(p^+, S) > r$

Double r until **Inv 3** is satisfied or **Inv 2** is tight

Point Insertion

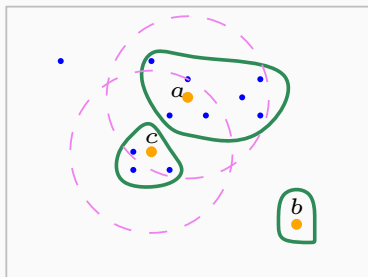
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Invariant 3: $\forall p \in P : \text{dist}(p, S) \leq 2r$

$k = 3$



$\text{dist}(a, b), \text{dist}(a, c), \text{dist}(b, c) \geq \frac{r}{2}$

$\text{dist}(p^+, S) > r$

Double r

Double r until **Inv 3** is satisfied or **Inv 2** is tight

Point Insertion

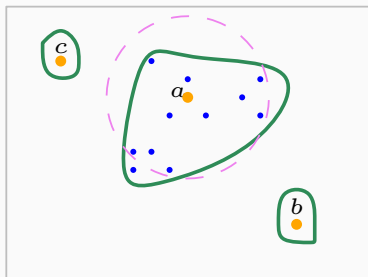
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$\text{dist}(p^+, S) \leq r$

Double r

Point Insertion

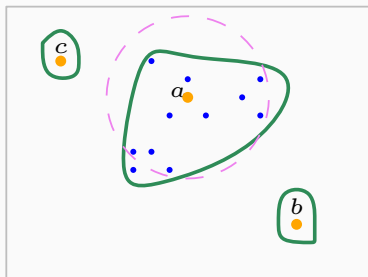
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$\text{dist}(p^+, S) \leq r$

Double r

Concern: Can the approximation ratio explode?

Point Insertion

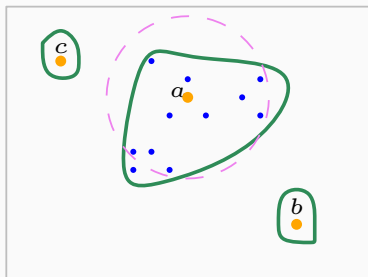
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$k = 3$



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$\text{dist}(p^+, S) \leq r$

Double r

Order using maximal distance- r independent set on centers